PMM U.S.S.R., Vol.47, No.6, pp. 775-782, 1983
Printed in Great Britain

0021-8928/83 $10.00+0.00

UDC 539.3

CERTAIN VERSIONS OF THE FORMULATION OF PROBLEMS OF NON-LINEAR ELASTICITY IN TERMS OF STRESSES*

S.Ia. Makovenko

New versions of the formulation of static non-linear elasticity theory problems are elucidated in terms of stresses for materials with locally reversible state laws (for instance, for a semilinear John material /1/), that reduce to the solution of nine equations in the Piola stress tensor component for six or three boundary conditions and three integral conditions. This paper is related to the investigations in /2—7/ devoted to analogous problems of the mechanics of a solid linearly deformable body. Examples are presented of the realization of one of the versions.

1. Traditional formulation of the problem in terms of stresses (Problem A). In a certain Lagrange system of coordinates let the defining relationships relating the Piola stress tensor $\mathbf{P}$ and the gradient of the position vector $\nabla\mathbf{R}$ be given in the form

$$P^{ij} = P^{ij}(\nabla\mathbf{R}) \quad (\mathbf{P} = \mathbf{P}(\nabla\mathbf{R}))$$

and also let the following reversible relationships hold

$$(\nabla\mathbf{R})_{ij} = C_{ij}(\mathbf{P}) \quad (\nabla\mathbf{R} = \mathbf{C}(\mathbf{P})) \tag{1.1}$$

For a semilinear John material /1/

$$C_{ij}(\mathbf{P}) = \frac{1}{2\mu}\left[P^{st}g_{si}g_{tj} + \left(2\mu - \frac{\nu}{1+\nu}f_1\right)b_{ij}\right] \tag{1.2}$$

where g_{st} are the metric tensor components of the undeformed medium, b_{ij} are the components of the tensor $(\mathbf{P}\cdot\mathbf{P}^T)^{-1/2}\cdot\mathbf{P}$, f_1 is the first invariant of the tensor $(\mathbf{P}\cdot\mathbf{P}^T)^{1/2}$, and μ, ν are constants of elasticity.

Let the following equations of statics be given

$$\nabla_i P^{ij} + K^j = 0 \quad (\nabla\cdot\mathbf{P} + \mathbf{K} = 0) \tag{1.3}$$

where $\mathbf{K}$ are given volume forces and boundary conditions of mixed type: the forces $\mathbf{f}\,dO/do$ or $\mathbf{f}_0$ ("dead" loads) are given on the part o_1 of the body boundary, while on the other part o_2 we are given the position vector $\mathbf{R}_0$

$$n_i P^{ij}|_{o_1} = f^j\frac{dO}{do} = f_0^{\,j}, \quad \chi_i|_{o_2} = \chi_i^\circ \tag{1.4}$$

In addition, in the case of the action of a dead load the Signorini integral compatibility condition must be satisfied /1/

$$\int_v \mathbf{R}\times\mathbf{K}\,dv + \int_o \mathbf{R}\times\mathbf{f}_0\,do = 0 \tag{1.5}$$

which expresses the fact that the principal moment of the external forces vanishes in the deformed state of the body. We will assume that all the functions introduced possess the smoothness needed to carry out the transformations employed. We shall also assume the presence of a "natural", i.e., unstressed, state of the initial undistorted configuration of the body. Moreover, unless otherwise stated, we will confine ourselves to the following dead loading:

$$\mathbf{K}\,dV = \mathbf{K}_0\,dv, \quad \mathbf{f}\,dO = \mathbf{f}_0\,do \tag{1.6}$$

and we will assume the dimensions of the elastic body to be finite.

If the volume v occupied by the body prior to deformation is a simply-connected domain, the necessary and sufficient conditions for the system of differential equations (1.1) to be integrable with respect to the component χ_i of the position vector $\mathbf{R}$ are the conditions that the non-symmetric tensor should vanish, viz.

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*Prikl.Matem.Mekhan., 47, 6, 972-980, 1983

$$M^{ij} = \varepsilon^{lsi} g^{tj} \nabla_l C_{st}(\mathbf{P}) = 0 \quad (\mathbf{M} = \nabla \times \mathbf{C}(\mathbf{P}) = 0) \tag{1.7}$$

In this case, the position vector $\mathbf{R}$ can be expressed in terms of the Piola stress tensor

$$\mathbf{R} = \mathbf{R}(q_0) + \int_{q_0}^{q} d\mathbf{r} \cdot \mathbf{C}(\mathbf{P}) \tag{1.8}$$

where $q\,q_0$ is the arc of integration with origin at an arbitrarily fixed point q_0, and boundary conditions (1.4) can be written in the form

$$n_i P^{ij}|_{o_1} = f_0{}^j \quad (\mathbf{n} \cdot \mathbf{P}|_{o_1} = \mathbf{f}_0) \tag{1.9}$$
$$\chi_i(\mathbf{P})|_{o_2} = \chi_i{}^0 \quad (\mathbf{R}(\mathbf{P})|_{o_2} = \mathbf{R}_0)$$

The traditional formulation of the static problem of the non-linear theory of elasticity in terms of stresses (Problem A) is given by relationships (1.3), (1.7), (1.9), and (1.5). As in the linear formulation, the problem in terms of stresses turns out to be overdefined: nine components of the non-symmetric Piola stress tensor should satisfy the 12 equations (1.3) and (1.7) in general complexity.

New modifications of the formulation of the problems in the non-linear theory of elasticity in terms of stresses, which are completely equivalent to the traditional formulation (Problem A) but free from the overdefinition properties of the system of governing equations are elucidated below.

2. Method of combination equations (Problem B). Following /2–4/, we set

$$a^j = \nabla_i P^{ij} + K^j \quad (\mathbf{a} = \nabla \cdot \mathbf{P} + \mathbf{K}) \tag{2.1}$$
$$b_j = t_{ji} a^i \quad (\mathbf{b} = \mathbf{T} \cdot \mathbf{a})$$

where $\mathbf{T}$ is an arbitrary non-degenerate tensor.

As $\mathbf{a} \to 0$ we evidently have $\mathbf{b} \to 0$ and conversely.

Let us form the combination equations

$$g^{nj}(g^{im} \nabla_i b_j + \varepsilon^{lsm} \nabla_l C_{sj}) = 0 \quad (\nabla \mathbf{b} + \nabla \times \mathbf{C} = 0) \tag{2.2}$$

Also, for points lying on the surface of an undeformed body, let

$$a^j|_o = 0 \quad (\mathbf{a}|_o = 0) \tag{2.3}$$

The new formulation of the problem in terms of stresses (Problem B) is given by (2.2), (1.9), (1.5) and (2.3): nine Piola stress tensor components should satisfy nine equations (2.2), six boundary conditions (1.9) and (2.3), and condition (1.5).

Theorem 1. Problem B is equivalent to Problem A.

We apply the operation div to (2.2). Taking into account that $\nabla \cdot \nabla \times \mathbf{C} \equiv 0$, as well as (2.3), we obtain

$$\nabla^2 \mathbf{b} \equiv 0, \quad \mathbf{b}|_o = 0 \tag{2.4}$$

When (2.4) is satisfied $\mathbf{b} \equiv 0$ everywhere in the domain v. It is sufficient to prove this assertion in some particular system of coordinates Cartesian, say. Then it will also be true in any other allowable system of coordinates because of the invariance of the tensor relations. But all the components of the vector $\mathbf{b}$ are harmonic functions in a Cartesian system of coordinates, equal to zero on the contour o, and therefore, are everywhere equal to zero in v because of the properties of harmonic functions. Consequently $a^i \equiv 0$ everywhere in v also. It now follows from (2.2) that $M^{ij} \equiv 0$. Thus, satisfaction of the conditions of Problem B results in identical satisfaction of the conditions of Problem A.

The converse assertion also holds. Indeed

$$\mathbf{a} = 0 \Rightarrow \mathbf{b} = 0 \Rightarrow \nabla \mathbf{b} = 0$$

and together with (1.7), (2.2) and (2.3) of Problem B are satisfied identically. The theorem is proved.

3. Method of weakened strain compatibility conditions (Problem B). Here and henceforth, let the domain v be a hexahedron bounded by the coordinate surfaces $q^i = c^i \pm h^i$, where $q^i = c^i$ are mean surfaces of the domain v, and h^i are arbitrary parameters characterizing the dimensions of the domain v. (An arbitrary simply-connected domain with smooth boundaries can always be inscribed in a coordinate hexahedron, and a function given in v can be predetermined therein in a continuous manner /8/).

Everywhere in the domain v let

$$M^{12} = M^{21} = M^{23} = M^{32} = M^{13} = M^{31} = 0 \tag{3.1}$$

and suppose the remaining components M^{11}, M^{22}, M^{33} vanish only an individual fixed coordinate surfaces, namely

$$M^{11} = 0 \qquad \text{on} \qquad q^1 = c^1 \quad (1, 2, 3) \tag{3.2}$$

A new modification of the formulation of the problem in terms of stresses (Problem B) is given by (1.3), (3.1), (1.9), (3.2), (1.5).

Theorem 2. Problem B is equivalent to Problem A.

It is obviously sufficient to prove that conditions (3.1), (3.2) are equivalent to conditions (1.7). Indeed, the components M^{ij} of the tensor $\mathbf{M} = \nabla \times \mathbf{C}(\mathbf{P})$ are only conditionally independent, being related by the three differential equations

$$\nabla_i M^{ij} \equiv 0 \quad (\nabla \cdot \mathbf{M} = \nabla \cdot \nabla \times \mathbf{C}(\mathbf{P}) \equiv 0) \tag{3.3}$$

Taking (3.1) into account we can convert (3.3) to the form

$$\partial_1 m^{11} + \Gamma_{11}{}^1 m^{11} + \Gamma_{22}{}^1 m^{22} + \Gamma_{33}{}^1 m^{33} \equiv 0 \quad (1, 2, 3) \tag{3.4}$$

$$\left(\partial_i \ldots \equiv \frac{\partial \ldots}{\partial q^i}, \quad m^{ii} = \sqrt{g}\, M^{ii}, \quad g = \det[g_{ij}] > 0\right)$$

where $\Gamma_{kl}{}^i$ are Christoffel symbols of the second kind.

According to conditions (3.2)

$$m^{11} = 0, \quad q^1 = c^1 \quad (1, 2, 3) \tag{3.5}$$

We have arrived at a homogeneous Cauchy problem (3.4) and (3.5) which has the obvious trivial solution

$$m^{11} \equiv m^{22} \equiv m^{33} \equiv 0 \tag{3.6}$$

everywhere in v.

If the coefficients $\Gamma_{jj}{}^i$ have the property of continuity in the domain v, the trivial solution (3.6) will be unique in v.

Let us prove this. To this end we replace (3.4) and (3.5) by an equivalent system of homogeneous Volterra integral equations of the second kind

$$m^{ii} + \int_{c^i}^{q^i} \sum_{j=1}^{3} \Gamma_{jj}{}^i m^{jj}\, d\xi^i \equiv 0 \tag{3.7}$$

We consider the first quadrant v_1 of the domain v:

$$v_1 = \{q^i : c^i \leqslant q^i \leqslant c^i + h^i, \ i = 1, 2, 3\}$$

We assume that together with the trivial solution (3.6), a non-trivial sufficiently smooth solution m^{ii} exists that also satisfies (3.7). In this case, at a certain point $q_0{}^i \in v_1$ the quantity

$$m = \alpha_1 |m^{11}| + \alpha_2 |m^{22}| + \alpha_3 |m^{33}|$$

($\alpha_i > 0$ are dimensional coefficients) takes the maximum value $m_0 > 0$.

Let us consider

$$m_0 = \sum_{i=1}^{3} \alpha_i \left| \int_{c^i}^{q_0{}^i} \sum_{j=1}^{3} \Gamma_{jj}{}^i m^{jj}\, d\xi^i \right| \leqslant m_0 h$$

$$\left(h = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\alpha_i h^j}{\alpha_j} \max_{v_1} |\Gamma_{jj}{}^i|\right)$$

Therefore, there should be $1 \leqslant h$ which is impossible because of the arbitrariness of h.

This contradiction proves the uniqueness of the trivial solution (3.6) in the first quadrant v_1.

The uniqueness of the solution (3.6) in the remaining quadrants of the domain v is proved analogously.

In the Cartesian coordinate system $\Gamma_{jj}{}^i \equiv 0 \Rightarrow m^{ii} \equiv 0$ everywhere in v, in agreement with (3.7), the domain v can here be unbounded also.

Thus, conditions (3.1) and (3.2) are equivalent to conditions (1.7). The theorem is proved.

Conditions (3.1) (fundamental) and (3.2) (additional) are one of the modifications of weakened strain compatibility conditions. We have just 27 such modifications equivalent to one other. Combinations of the components M^{ij} corresponding to an additional group of conditions of certain modifications are presented in the table. The missing combinations are obtained from those in the table by cyclic permutation of the superscripts. Six of the nine components of M^{ij} not in the additional group are in the fundamental group of conditions of the same modifications.

It can be shown in the same way that all the weakened strain compatibility conditions obtained in this manner are equivalent to conditions (1.7).

№	$q^1 = c^1$	$q^2 = c^2$	$q^3 = c^3$
1	M^{21}	M^{32}	M^{13}
2	M^{31}	M^{12}	M^{23}
3	M^{11}	M^{22}	M^{33}
4	M^{21}, M^{31}	—	M^{13}
5	M^{21}	—	M^{13}, M^{33}
6	M^{31}	—	M^{13}, M^{23}
7	—	—	M^{13}, M^{23}, M^{33}
8	M^{31}	M^{22}	M^{13}
9	—	M^{22}	M^{13}, M^{33}
10	—	M^{12}	M^{23}, M^{33}
11	—	M^{12}, M^{22}	M^{33}

4. Method of integro-differential strain compatibility conditions (Problem Γ).

Consider the relations

$$\partial_i \chi_j - \Gamma_{ij}{}^k \chi_k = C_{ij}(\mathbf{P}) \quad (\nabla \mathbf{R} = \mathbf{C}(\mathbf{P})) \tag{4.1}$$

We take any three of the nine equations, which is a system integrable with respect to χ_i

$$\partial_{s_m} \chi_m - \Gamma^k_{s_m m} \chi_k = C_{s_m m}(\mathbf{P}) \quad (m = 1, 2, 3; \forall s_m \in \{1, 2, 3\}) \tag{4.2}$$

which contains 27 different modifications depending on the combinations of values of the subscripts s_m.

We integrate (4.2) with respect to χ_m, or equivalently, we find the solution of a system of integral equations of the form

$$\chi_m - l_m{}^n \chi_n = \psi_m + \zeta_m \tag{4.3}$$

$$\left(l_m{}^n \ldots = \int\limits_{c^{s_m}}^{q^{s_m}} \Gamma^n_{s_m m} \ldots d\xi^{s_m}, \quad \psi_m = \int\limits_{c^{s_m}}^{q^{s_m}} C_{s_m m}(\mathbf{P})\, d\xi^{s_m} \right)$$

where $\zeta_m = \chi_m |_{q^{s_m} = c^{s_m}}$ are arbitrary two-dimensional functions defined on a fixed surface $q^{s_m} = c^{s_m}$.

We represent (4.3) as the operator equation

$$(\mathbf{I} - \mathbf{L}) \cdot \mathbf{R} = \boldsymbol{\psi} + \boldsymbol{\zeta} \tag{4.4}$$

$$\mathbf{R} = \operatorname{col}(\chi_1, \chi_2, \chi_3), \quad \boldsymbol{\psi} = \operatorname{col}(\psi_1, \psi_2, \psi_3)$$

$$\boldsymbol{\zeta} = \operatorname{col}(\zeta_1, \zeta_2, \zeta_3), \quad \mathbf{L} \ldots = \| l_m{}^n \ldots \|_{n, m = 1, 2, 3}$$

The space of continuous functions in the domain v is denoted by C_v by introducing the norm

$$\| u \|_v = \max_v | u |$$

in this set.

Correspondingly, we define the vector functional space $\mathbf{C}_v$ with the norm

$$\| \mathbf{u} \|_v = \sum_{i=1}^{3} \| u_i \|_v$$

We set

$$\Gamma_{mn}{}^i \in C_v, \quad \mathbf{R}, \boldsymbol{\psi}, v \in \mathbf{C}\zeta$$

in the problem under consideration.

If the domain of definition of the operator $\mathbf{L}$ is the set $\mathbf{C}_v$, the domain of values will evidently be the set of elements $\boldsymbol{\alpha} = \mathbf{L} \cdot \mathbf{R} \in \mathbf{C}_v$, i.e., the operator $\mathbf{L}$ maps $\mathbf{C}_v$ into itself.

It can be confirmed that the operator $\mathbf{L}$ is linear and continuous in $\mathbf{C}_v$, and its norm allows the obvious estimate.

$$\| \mathbf{L} \|_v \leqslant q = \sum_{m=1}^{3} \sum_{n=1}^{3} \| \Gamma^m_{s_n n} \|_v h^m$$

from which it can be seen that the norm depends on the dimensions of the domain v. We constrain these dimensions in such a way that the following condition is satisfied:

$$q < 1 \tag{4.5}$$

Condition (4.5) ensures the existence of the operator $(\mathbf{I}-\mathbf{L})^{-1}=\mathbf{B}$ which is the inverse of the operator $\mathbf{I}-\mathbf{L}$ and is representable in the form /9/

$$\mathbf{B}=\sum_{k=0}^{\infty}\mathbf{L}^k \tag{4.6}$$

where the operator $\mathbf{B}$, like $\mathbf{L}$, is also linear and continuous in $\mathbf{C}_v$.

The existence of the inverse operator $\mathbf{B}$ ensures that the solution of the operator equation (4.4) is unique, that it depends continuously on the given vector function $\boldsymbol{\psi}$ and the arbitrary function $\boldsymbol{\zeta}$, and that it can be represented in conformity with (4.6) in the form

$$\mathbf{R}=\mathbf{B}\cdot(\boldsymbol{\psi}+\boldsymbol{\zeta})=\sum_{k=0}^{\infty}\mathbf{L}^k\cdot(\boldsymbol{\psi}+\boldsymbol{\zeta})$$

or written in component-by-component form

$$\chi_m = L_m{}^n(\psi_n+\zeta_n);\qquad (L_m{}^n\ldots=\delta_m{}^n\ldots+l_m{}^n\ldots+l_m{}^i l_i{}^n\ldots+l_m{}^i l_i{}^j l^n{}_j\ldots+\ldots) \tag{4.7}$$

Substituting (4.7) into (4.1), we obtain the strain continuity condition in terms of stresses in the integro-differential form

$$(\partial_i L_j{}^n\ldots-\Gamma_{ij}{}^k L_k{}^n\ldots)(\psi_n+\zeta_n)=C_{ij}(\mathbf{P})\qquad ((i,\ j)\neq(s_m,\ m)) \tag{4.8}$$

Conditions (4.8) can obviously be interpreted as the conditions for (4.1) to be solvable in the form (4.7).

The new modification of the problem in non-linear elasticity theory in terms of stresses (Problem Γ) is given by (1.3), (4.8), (1.9) and (1.5).

Theorem 3. Problem Γ is equivalent to Problem A.

The equivalence of conditions (4.8) and conditions (1.7) must be proved as well as the identity of representations (1.8) and (4.7) of the position vector $\mathbf{R}(\mathbf{P})$. We temporarily denote the vectors (1.8) and (4.7) by $\mathbf{R}_A$ and $\mathbf{R}_\Gamma$, respectively.

Suppose we know that conditions (4.8) are satisfied, i.e., $\nabla\mathbf{R}_\Gamma\equiv\mathbf{C}(\mathbf{P})$. Then

$$\nabla\times\nabla\mathbf{R}_\Gamma\equiv\nabla\times\mathbf{C}(\mathbf{P})\equiv 0.$$

Setting $q_0=(c^1,\ c^2,\ c^3)$ we have

$$\mathbf{R}_A\equiv\mathbf{R}(q_0)+\int_{q_0}^{q}d\mathbf{r}\cdot\nabla\mathbf{R}_\Gamma\equiv\mathbf{R}_\Gamma$$

which it was required to prove.

Theorem 4. The strain integro-differential compatibility conditions of the form (4.8) are a corollary of the variational principle of the stationarity of complementary work.

According to the principle of the stationarity of complementary work /10/, the quantity called complementary work has a stationary value in an actually realizable equilibrium state, written in the form

$$\int_v \delta\mathbf{P}\cdot\cdot\mathbf{C}^T(\mathbf{P})\,dv-\int_{o_1}\mathbf{n}\cdot\delta\mathbf{P}\cdot\mathbf{R}_0\,do=0 \tag{4.9}$$

Statically possible states of stress are subjected to comparison, consequently

$$\nabla\cdot\delta\mathbf{P}=0,\quad \mathbf{n}\cdot\delta\mathbf{P}=\begin{cases}0 & \text{on}\quad o_2\\ \delta\mathbf{f}_0 & \text{on}\quad o_1\end{cases} \tag{4.10}$$

Relying on conditions (4.10) and on the well-known formula for converting a volume into a surface integral, we can verify the validity of the relationship

$$-\int_v(\delta\mathbf{P}_1\cdot\cdot(\mathbf{C}^T(\mathbf{P})-\nabla\mathbf{R}_\Gamma^T)+\delta\mathbf{P}\cdot\cdot\nabla\mathbf{R}_\Gamma^T)\,dv+\int_{o_1}\delta\mathbf{f}_0\cdot\mathbf{R}_\Gamma\,do\equiv 0 \tag{4.11}$$

$$\left(\delta\mathbf{P}_1=\sum_{i=1}^{3}\mathbf{r}_{s_i}\mathbf{r}_i\delta P^{s_i i}\right)$$

where $\mathbf{r}_i$ is the vector basis of the v-configuration.

Combining (4.9) and (4.11) and equating the factors in the integrands to zero for arbitrary and independent variations $\delta\mathbf{P} - \delta\mathbf{P}_1$ and $\delta\mathbf{f}_0$, we arrive at the strain continuity condition $\mathbf{C}(\mathbf{P}) - \nabla\mathbf{R}_\Gamma = 0$ and the kinematic boundary condition $(\mathbf{R}_\Gamma - \mathbf{R}_0)|_{o_1} = 0$. The theorem is proved.

5. **Examples of representations of Problem Γ.** *Axisymmetric strain of a solid of revolution.* We select the cylindrical coordinates

$$c^1(q^3) \leqslant q^1 \leqslant d^1(q^3),\ 0 \leqslant q^2 \leqslant 2\pi,\ c^3 \leqslant q^3 \leqslant d^3$$

as material coordinates by superposing the q^3 axis on the axis of the solid of revolution.

In the case under consideration

$$p^{ij} = p^{ij}(q^1, q^3),\ p^{12} = p^{21} = p^{23} = p^{32} = 0 \tag{5.1}$$

In conformity with (1.2), we obtain

$$C_{11} = \frac{p^{11}}{2\mu} + \delta\cos\alpha,\quad C_{22} = \left(g_{22}\frac{p^{22}}{2\mu} + \delta\right)g_{22} \tag{5.2}$$

$$C_{33} = \frac{p^{33}}{2\mu} + \delta\cos\alpha,\quad C_{13} = \frac{p^{13}}{2\mu} + \delta\sin\alpha$$

$$C_{31} = \frac{p^{31}}{2\mu} - \delta\sin\alpha,\quad C_{12} = C_{21} = C_{32} = C_{23} = 0$$

$$\delta = 1 - \frac{\nu}{2\mu(1+\nu)}(q + p^{22}g_{22})$$

$$q = [(p^{11} + p^{33})^2 + (p^{13} - p^{31})^2]^{1/2}$$

$$g_{22} = {q^1}^2,\quad \cos\alpha = \frac{p^{11} + p^{33}}{q},\quad \sin\alpha = \frac{p^{13} - p^{31}}{q}$$

The equilibrium equations (1.3) and the strain continuity (4.8) for $s_1 = s_2 = s_3 = 3$ are transformed to the form

$$l_1 p^{11} + \partial_3 p^{31} - q^1 p^{22} + K^1 = 0 \tag{5.3}$$

$$\partial_3 p^{33} + l_1 p^{13} + K^3 = 0\quad \left(l_1 \ldots \equiv \partial_1 \ldots + \frac{1\ldots}{q^1}\right)$$

$$\partial_1\chi_1 = C_{11},\ \partial_1\chi_3 = C_{13},\ q^1\chi = C_{22} \tag{5.4}$$

where

$$\chi_1 = \zeta_1 + \int_{c^3}^{q^3} C_{31}\,d\xi^3,\quad \chi_2 = 0,\quad \chi_3 = \zeta_3 + \int_{c^3}^{q^3} C_{33}\,d\xi^3 \tag{5.5}$$

When the surface forces $\mathbf{f}(\mathbf{f}_0)$ are given we append the boundary conditions

$$n_i p^{ij}|_o = f^j\frac{dO}{do}\,(= f_0{}^j)\quad (i, j = 1,3) \tag{5.6}$$

to equations (5.3) and (5.4).

If the state of stress is not accompanied by rotations $(\alpha = 0)$, and there are no mass forces $(K^1 = K^3 = 0)$, the solution of the system of equations (5.3) and (5.4) can be expressed in terms of the harmonic function ψ

$$\frac{p^{11}}{2\mu} = \partial_1{}^2\psi - \frac{b}{g_{22}} - C_2,\quad \frac{p^{22}}{2\mu} = \frac{1}{g_{22}}\left(\frac{1}{q^1}\partial_1\psi + \frac{b}{g_{22}} - C_2\right)$$

$$\frac{p^{33}}{2\mu} = \partial_3{}^2\psi + d,\quad \frac{p^{13}}{2\mu} = \frac{p^{31}}{2\mu} = \partial_1\partial_3\psi$$

$$\chi_1 = \partial_1\psi + \frac{b}{q^1},\quad \chi_3 = \partial_3\psi + C_1(q^3 - c^3) + a$$

$$\left(C_1 = \frac{1 + \nu + (1 - 2\nu)d}{1 - \nu},\quad C_2 = \frac{1 + \nu - \nu d}{1 - \nu},\right.$$

$$a,\ b,\ d = \text{const})$$

In particular, setting $\psi = t(q^3 - c^3) - aq^3 - tg_{22}/2\ (t = \text{const})$, we arrive at the well-known solution obtained for a hollow cylinder by the method of displacements /1/.

Plane strain of a prismatic body. We select the Cartesian coordinates q^1, q^2, q^3 of the reference configuration $(c^i \leqslant q^i \leqslant d^i)$ as material coordinates.

Keeping the same scheme of representing the Problem Γ as in the preceding example, we arrive at relationships that are completely analogous to (5.1)–(5.6) with the sole difference that now it should be considered everywhere that $g_{22} = 1,\ l_1\ldots \equiv \partial_1\ldots$, the term $q^1 p^{22}$ should be discarded in the first equilibrium equation, the third equation in (5.4) should be replaced by $C_{22} = c = \text{const}$, and we put $\chi_2 = cq^2$ in (5.5).

The solution of (5.3) and (5.4) is represented in the case under consideration when there are no mass forces by

$$\frac{p^{11}}{2\mu}=\partial_3\Phi,\quad \frac{p^{33}}{2\mu}=-\partial_1 f,\quad \frac{p^{13}}{2\mu}=\partial^3 f,\quad \frac{p}{2\mu}=-\partial_1\Phi$$
$$\frac{p^{22}}{2\mu}=\frac{c-1-\nu+\nu q^*}{1-\nu},\quad q^*=[(\partial_3{}^2\psi)^2+(\partial_1\partial_3\psi)^2]^{1/2}$$
$$\chi_1=\partial_1\psi+f,\quad \chi_3=-\partial_3\psi-\Phi$$

where ψ is a harmonic function and the functions Φ and f should satisfy the equations

$$\partial_3\Phi-\partial_1 f=q\cos\alpha/2\mu \tag{5.7}$$
$$\partial_1\Phi+\partial_3 f=q\sin\alpha/2\mu$$
$$\left(\frac{q}{2\mu}=\frac{\nu c-1-\nu+q^*}{1-\nu},\quad \cos\alpha=-\frac{\partial_3{}^2\psi}{q^*},\quad \sin\alpha=-\frac{\partial_1\partial_3\psi}{q^*}\right)$$

System (5.7) obtained is identical with the corresponding complex equation of the method of displacements /1/.

Three-dimensional strain of thick slabs. We refer the medium occupied by the slab to a Cartesian system of coordinates $(q^i\equiv x_i)$. The equilibrium (1.3), and the strain continuity equations (4.8) for $s_m=3, c^{s_m}=0$, the displacement formulas (4.7), and the equations of state for the semilinear material form the initial system

$$\partial_i p^{ij}+K_j=0,\ C_{mj}=\partial_m\chi_j \tag{5.8}$$
$$\chi_j=\zeta_j+\int_0^{x_3}C_{3j}\,d\xi_3\quad (m=1,2;\ j=1,2,3)$$
$$p^{ij}=\frac{2\mu}{1-2\nu}[(1-2\nu)C_{ij}+(\nu C-1-\nu)\delta_{ij}+F_{ij}]$$
$$F_{ij}=(1+\nu)(\delta_{ij}-a_{ij})+\nu(J_1 a_{ij}-C\delta_{ij})$$
$$C=C_{11}+C_{22}+C_{33}$$

where δ_{ij} is the Kronecker delta, J_1 is the first invariant of the tensor $(\mathbf{C}\cdot\mathbf{C}^T)^{1/2}$, a_{ij} are components of the rotation tensor $(\mathbf{C}\cdot\mathbf{C}^T)^{-1/2}\cdot\mathbf{C}$, and $\zeta_j=\zeta_j(x_1,x_2)$ are arbitrary two-dimensional functions.

In the case of conservation of the principal directions of the tensor of the Cauchy strain measures $a_{ij}\equiv\delta_{ij}$, $C\equiv J_1$, and $F_{ij}\equiv 0$. Hence, the functions F_{ij} are components of the correcting tensor.

We will determine F_{ij} by the method of successive approximations, thereby linearizing the problem.

In turn, we will solve the linear problem by the method of "initial functions" by reducing the three-dimensional problem to a two-dimensional one.

Using Lur'e's symbolic method /11/, we introduce the following notation for the operators

$$\partial_1\ldots\equiv\alpha,\ \partial_2\ldots\equiv\beta,\ \partial_1{}^2\ldots+\partial_2{}^2\ldots\equiv\gamma^2.$$

Let

$$P_j=\alpha F_{1j}+\beta F_{2j}+\partial_3 F_{3j}+K_j(1-2\nu)/(2\mu)$$

be the reduced volume load, and C_{31}, C_{32}, C_{33} resolving functions. We convert the initial equations (5.8) to a system of three governing equations

$$\int_0^{x_3}[(1-\nu)\gamma^2C_{31}+\nu\beta(\alpha C_{32}-\beta C_{31})]\,d\xi_3+\nu\alpha(C_{33}+\beta\zeta_2)+ \tag{5.9}$$
$$(1-2\nu)\partial_3C_{31}+[(1-\nu)\gamma^2-\nu\beta^2]\zeta_1+P_1=0\quad (1,2),\quad (\alpha,\beta)$$
$$(1-2\nu)\int_0^{x_3}\gamma^2C_{33}\,d\xi_3+\nu(\alpha C_{31}+\beta C_{32})+(1-\nu)\partial_3C_{33}+(1-2\nu)\gamma^2\zeta_3+P_3=0$$

We introduce the following transcendental operators

$$\omega_1=\sin\gamma x_3,\quad \omega_2=\cos\gamma x_3$$
$$\omega_3=\frac{1}{\gamma}(\omega_1+x_3\gamma\omega_2)$$
$$i_1(\ldots)=\int_0^{x_3}(x_3-\xi_3)\sin\gamma(x_2-\xi_3)(\ldots)\,d\xi_3$$
$$i_2(\ldots)=\int_0^{x_3}(x_3-\xi_3)\cos\gamma(x_3-\xi_3)(\ldots)\,d\xi_3$$
$$i_3(\ldots)=\int_0^{x_3}\sin\gamma(x_3-\xi_3)(\ldots)\,d\xi_3$$

$$i_4(\ldots) = \int_0^{x_3} \cos\gamma(x_3 - \xi_3)(\ldots)\, d\xi_3$$

The particular solution of the homogeneous system of equations (5.9)(for $P_j = 0$) has the form

$$C_{31}^0 = \omega_3\sigma_1 - \gamma\omega_1\zeta_1 - \frac{\nu}{2(1-2\nu)}\alpha\omega_3\zeta - \frac{\nu x_3\alpha\omega_1\sigma}{2(1-\nu)\gamma},$$
$$C_{32}^0 = \omega_3\sigma_2 - \gamma\omega_1\zeta_2 - \frac{\nu\beta\omega_3\zeta}{2(1-2\nu)} - \frac{\nu x_3\beta\omega_1\sigma}{2(1-\nu)\gamma}$$
$$C_{33}^0 = \omega_3\sigma_3 - \gamma\omega_1\zeta_3 - \frac{\nu\omega_3\sigma}{2(1-\nu)} + \frac{\nu x_3\gamma\omega_1\zeta}{2(1-2\nu)}$$
$$(\sigma = \sigma_3 + \alpha\zeta_1 + \beta\zeta_2, \quad \zeta = \alpha\sigma_1 + \beta\sigma_2 - \gamma^2\zeta_3)$$

where $\sigma_j \equiv C_{3j}|_{x_3=0}$ are arbitrary two-dimensional functions.

Now setting $\sigma_j = \zeta_j = 0$, we arrive at the particular solution of the inhomogeneous system (5.9)

$$(1-2\nu)C_{31}^* = -i_4(P_1) + \frac{\nu\alpha}{2(1-\nu)\gamma}[i_1(P) + \gamma i_2(P_3) + i_3(P_3)]$$
$$(1-2\nu)C_{32}^* = -i_4(P_2) + \frac{\nu\beta}{2(1-\nu)\gamma}[i_1(P) + \gamma i_2(P_3) + i_3(P_3)]$$
$$(1-\nu)C_{33}^* = -i_4(P_3) + \frac{\nu}{2(1-2\nu)}\left[-\gamma i_1(P_3) + i_2(P) + \frac{1}{\gamma}i_3(P)\right]$$
$$(P = \alpha P_1 + \beta P_2).$$

The two-dimensional functions ζ_j, σ_j (the initial functions) are determined from the boundary conditions $p^{3j} = f_0^j$ or $\chi_j = \chi_j^0$ on $x_3 = 0, h_3$.

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Translated by M.D.F.